DFT and Persistent Homology for Topological Musical Data Analysis

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Abstract. There are several works that already exist in the context of persistent homology for Topological Musical Data Analysis, and we can cite [2] and [3] among others: in each one of these works, the main problem is to find how we can associate a point cloud with a musical score, that is a set of points with a metric. This paper proposes to combine persistent homology with a symbolic representation of musical structures given by the Discrete Fourier Transform to answer this question: the points are the musical bars and the metric is given by the DFT in dimension two. We start with the mathematical background, and the main goal of this paper is thus to support the use of the DFT in this context, by extracting barcodes from artificially constructed scores based on Tonnetze, and then recovering topological features.

Keywords: Filtered simplicial complex · Persistent homology · Barcodes · Discrete Fourier Transform · Topological Data Analysis · Musical analysis · Tonnetz.

1 Context, Definitions and Problematic

1.1 Persistent Homology

A finite filtered simplicial complex $K = (K^i)_{i=0}^N$ is a nested sequence of simplicial complexes, such as

$$\emptyset = K^{-1} \subset K^0 \subseteq K^1 \subseteq \ldots \subseteq K^N = K$$

We call $\{0, 1, \ldots, N\}$ the set of **filtration times**. For instance, Figure 1 represents a filtered complex with six filtration times.

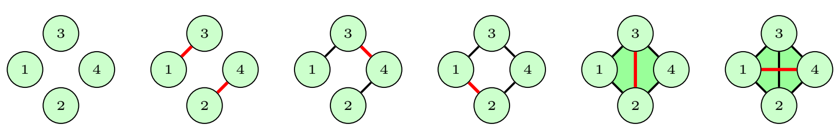


Fig. 1: A filtered simplicial complex K with six times of filtration.

For any filtered complex K and any pair of integers $(i, p) \in \mathbb{N}^2$, there exists a natural inclusion map $\eta^{i,p} : K^i \hookrightarrow K^{i+p}$ which induces a homological morphism

$$\eta_*^{i,p} : H_*(K^i) \to H_*(K^{i+p})$$

Therefore, for each degree d, we get the following sequence of dth-homology groups:

$$0 \longrightarrow H_d(K^0) \xrightarrow{\eta_d^{0,1}} H_d(K^1) \xrightarrow{\eta_d^{1,1}} ...H_d(K^{N-1}) \xrightarrow{\eta_d^{N-1,1}} H_d(K^N) = H_d(K)$$

Each one of the maps $\eta_d^{i,p}$ sends a d-homological class at time i to the one containing it at time $i+p$ of the filtration. If the cycle has become a boundary, then it sends the class to 0. Thus, the image of each map $\eta_*^{i,p}$ will describe the homology evolution during the filtration: more precisely, the **p-persistent homology group** associated with K^i in degree d is given by the image of each map $\eta_*^{i,p}$:

$$H_d^{i,p}(K, \mathbb{F}_2) = H_d^{i,p}(K) := \operatorname{im} \eta_d^{i,p} : K^i \to K^{i+p}.$$

From [12], we have the following structure theorem of persistent homology:

Theorem 1. *Let K be a finite filtered simplicial complex. In any degree n, we have the following isomorphism of $\mathbb{F}_2[t]$-module:*

$$\bigoplus_{i=0}^{N} H_n(K^i) \cong \bigoplus_{j=0}^{N} t^{b_j} \, . \, \mathbb{F}_2[t] \big/ (t^{c_j})$$

This isomorphism naturally leads to the construction of graphics named barcodes, which mesures the lifetime evolution of homology classes in a filtration and provide a natural visualization of persistent homology.

Definition 1. *Let K be a finite filtered complex and $d \in \mathbb{N}$ be a fixed degree. The* ***barcode*** *$BC_d(K)$ associated with $H_d(K)$ is the graph where the x-axis describes the time of filtration and each generator of $H_d(K)$ corresponds to a bar whose start and end are given by Theorem's equivalence 1:*

- *a class that was born at time a_i and never dies is a bar that starts at the abscissa point a_i and never stops*
- *a class that was born at time b_j and died at time $b_j + c_j$ is a bar that starts and ends at abscissa points b_j and $b_j + c_j$, respectively.*

An illustration of this definition is given in Figure 2, where we have computed the corresponding barcodes family with filtration from Figure 1.

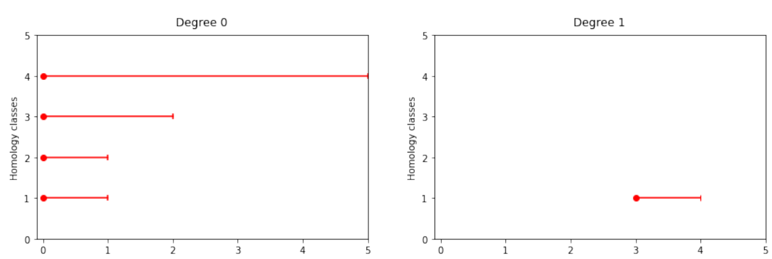


Fig. 2: The associated barcodes with filtration from Figure 1.

1.2 Topological Data Analysis

Persistent homology is an algebraic tool that is commonly used in Topological Data Analysis context. In practice, we have a starting object that we want to analyze using its hidden topological structure. For this purpose, we transform it into a **point cloud**, that is to say into a finite metric set, and from this point cloud, we build a filtered simplicial complex using the Vietoris-Rips algorithm, which is presented in Definition 2 below. Once this is done, it is possible to compute persistent homology and then extract the associated barcodes family. Therefore, we can use this family as a **topological fingerprint** to characterize the starting object. In our case, our starting object is given by a score, symbolically represented by a MIDI file. The general TDA process is summarized in Figure 3.

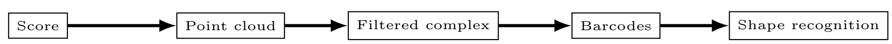


Fig. 3: Topological Data Analysis process.

Definition 2. *Let $X = \{x_1, ..., x_N\}$ be a point cloud, that means a collection of points in a metric space, and let $\epsilon \geq 0$ be a non-negative parameter. The* ***Vietoris-Rips complex*** *$\mathcal{R}_\epsilon(X)$ is the simplicial complex where :*

- ❋ *X is the set of vertices*
- ❋ *$\sigma = \{x_1, ..., x_n\}$ is a n-simplex if and only if the vertices it contains are pairwise close, i.e. if $d(x_i, x_j) \leq \epsilon$ for all pairs x_i, x_j of σ.*

Figure 4 shows the classical construction of a Vietoris-Rips complex starting from a given point cloud X and a parameter ϵ. The filtration is obtained by changing the parameter ϵ. In other words, this process consists of adding n-simplices to the point cloud as a certain parameter increases. The smaller the parameter is, the more separated the points are, while conversely, the larger the parameter is, the more trivial and topologically equivalent to a single point the resulting complex is. The point is therefore to analyze the complex for the "right" parameter, i.e. when the topological features are correctly represented. This construction method is presented in the article [7] by R. Ghrist.

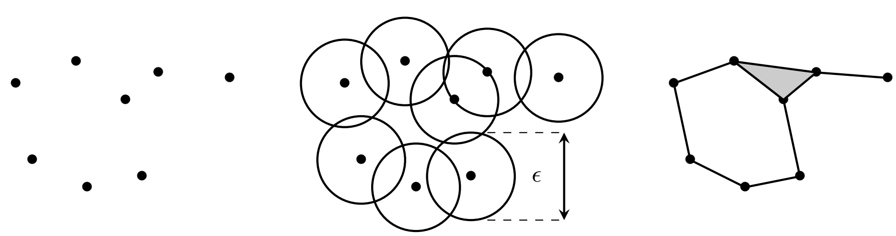


Fig. 4: The Vietoris-Rips method.

1.3 General Problematic

The essential question that remains incomplete in the TDA process is how to extract a point cloud from a musical score. There are many arbitrary choices to be made, and we already tried to answer this question in [4] by considering that a musical score is given by the set of its musical bars. Then, each bar was a subset of $\mathbb{R}^3$, so the metric was given by the Hausdorff distance. In this paper, we propose to keep the representation of a musical score by the set of its musical bars, but we consider the metric given by the two-dimensional Discrete Fourier Transform. In fact, we start by making explicit the transition from a musical bar to a subset of $\mathbb{Z}/t\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and then define the notion of DFT-distance on the set of musical bars. This construction allows us to obtain a filtered complex from a MIDI file, and then to extract a barcodes family using persistent homology. The second part focuses on the question of musical analysis using this new topological signature. In this paper, our main motivation is to support the use of the DFT in the context of persistent homology and TDA, so we propose an experiment on different famous musical structures from which we expect to capture topological features. More precisely, the point cloud will be extracted from sets of chords based on two-dimensional Tonnetze.

2 Persistent Homology Using Discrete Fourier Transform

Associating a filtered complex with a piece of music requires extracting a point cloud, i.e. a collection of points in a metric space. As mentioned in the previous paragraphs, the points are given by the musical bars of a score, and we propose here to use the two-dimensional Discrete Fourier Transform as the metric.

2.1 A Model Based on the Two-Dimensional DFT

The DFT is a mathematical tool introduced into the context of music analysis by D. Lewin in 1959, and used mainly for modeling simple musical sequences, such as scales or collections of rhythms (see the work of E. Amiot and J. Yust, for instance [1] and [10]).

In this paper, we are primarily interested in modeling a musical sequence using the DFT: if $\mathcal{M} = (m_1, \ldots, m_N)$ is a musical sequence in $\mathbb{Z}/n\mathbb{Z}$ (for instance a sequence of pitch-classes or rhythmic data), then it can be associated with a list of n Fourier coefficients $\big(\mathcal{F}_{\mathcal{M}}(0), \ldots, \mathcal{F}_{\mathcal{M}}(n-1)\big)$ using the corresponding characteristic map. For instance, the pitch-classes set from Figure 5 is given by $\mathcal{P} = \{9, 4, 5, 2\} \in \mathbb{Z}/12\mathbb{Z}$, and its characteristic map is given by

$$\mathbb{1}_{\mathcal{P}} : \{9, 4, 5, 2\} \mapsto (0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0).$$

In the same way, using the quaver note as the unit of time, $\mathcal{T} = \{0, 1, 3, 5\}$ becomes the set of **onsets**, and the associated characteristics map is given by

$$\mathbb{1}_{\mathcal{T}} : \{0, 1, 3, 5\} \mapsto (1, 1, 0, 1, 0, 1, 0, 0)$$

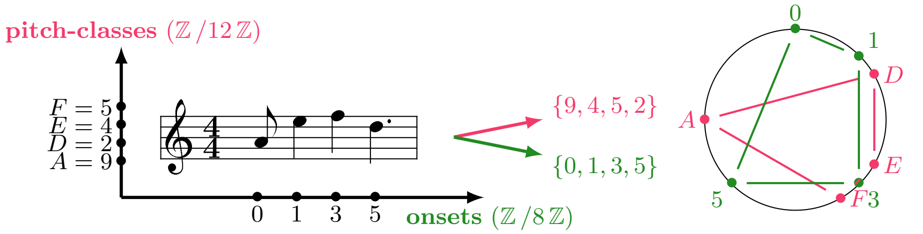


Fig. 5: A musical bar where the x-axis and y-axis represent time and pitches, respectively.

Now we can compute the **Discrete Fourier Transform** (DFT) of $\mathcal{M}$ ($\mathcal{P}$ or $\mathcal{T}$), which is simply given by the DFT of its characteristic map $\mathbb{1}_{\mathcal{M}} : \mathbb{Z}/n\mathbb{Z} \to \{0,1\}$ with the usual definition, that is:

$$\begin{aligned} \mathcal{F}_{\mathcal{M}} = \widehat{\mathbb{1}_{\mathcal{M}}} : \mathbb{Z}/n\mathbb{Z} &\longrightarrow \mathbb{C} \\ x &\longmapsto \sum_{k\in\mathcal{M}} \exp\left(\tfrac{-2i\pi kx}{n}\right) \end{aligned}$$

Therefore, each musical structure can be associated with a family of **Fourier coefficients** that are given by the n-tuple of the DFT values on $\mathbb{Z}/n\mathbb{Z}$:

$$\Big(\mathcal{F}_{\mathcal{M}}(0), \mathcal{F}_{\mathcal{M}}(1), \ldots, \mathcal{F}_{\mathcal{M}}(n-1)\Big)$$

The DFT can be applied to any abelian group, so the previous definition can naturally be generalized ot the two-dimensional case. More precisely, instead of considering onsets and pitch-classes separately, we combine both representations and thus see $\mathcal{T}$ and $\mathcal{P}$ as a subset of $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$, and we will call it a **musical bar**, denoted by $\mathcal{B}$. For instance, the description of the musical bar from Figure 5 is given by:

$$\mathcal{B} = (\mathcal{T}, \mathcal{P}) = \{(0,9), (1,4), (3,5), (5,2)\} \subset \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$$

In general, if t is a **unit of time** and p is an **ambitus**, both fixed, we say that a **musical bar** $\mathcal{B}$ with n elements is a subset of $\mathbb{Z}/t\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ given by

$$\mathcal{B} = \{(t_1, p_1), \ldots, (t_n, p_n)\}$$

where an element (t_j, p_j) of $\mathcal{B}$ is called a **note** which is characterized by its **onset** t_j modulo t and its **pitch-class** p_j modulo p. Therefore, a **musical score** $\mathcal{S}$ is the non-ordered set of its N distinct musical bars modulo (t,p):

$$\mathcal{S} = \{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_N\} \text{ with } \mathcal{B}_i \subset \mathbb{Z}/t\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \text{ and } \mathcal{B}_i \neq \mathcal{B}_j \text{ if } i \neq j$$

Thus, we get a description of any musical bar of a given score using a matrix of Fourier coefficients.

Definition 3. *Let B be a musical bar in $\mathbb{Z}/t\mathbb{Z}\times\mathbb{Z}/p\mathbb{Z}$.*

1. *The associated* ***characteristic map*** $\mathbb{1}_{\mathcal{B}} : \mathbb{Z}/t\mathbb{Z}\times\mathbb{Z}/p\mathbb{Z}$ *is given by:*

$$\mathbb{1}_{\mathcal{B}} : (x,y) \mapsto \begin{cases} 1 \text{ if } (x,y)\in\mathcal{B} \\ 0 \text{ otherwise.} \end{cases}$$

2. *The associated* ***DFT*** *is the DFT of the characteristic map* $\mathbb{1}_{\mathcal{B}}$*:*

$$\begin{aligned} \mathcal{F}_{\mathcal{B}} = \widehat{\mathbb{1}_{\mathcal{B}}} : \mathbb{Z}/t\mathbb{Z}\times\mathbb{Z}/p\mathbb{Z} &\longrightarrow \mathbb{C} \\ (x,y) \qquad &\longmapsto \sum_{(k,l)\in\mathcal{B}} \exp\left(\frac{-2i\pi kx}{t}\right)\exp\left(\frac{-2i\pi ly}{p}\right) \end{aligned}$$

3. *The associated* ***Fourier coefficients*** *are given by the matrix* $\widehat{M}_{\mathcal{B}} \in \mathcal{M}_{t\times p}(\mathbb{C})$*:*

$$\widehat{M}_{\mathcal{B}} = \Big(\mathcal{F}_{\mathcal{B}}(x,y)\Big)_{(x,y)\in\mathbb{Z}/t\mathbb{Z}\times\mathbb{Z}/p\mathbb{Z}}$$

2.2 From the DFT to a Point Coud

We denote by $\mathfrak{B}_{t,p}$ the set of all the musical bars in $\mathbb{Z}/t\mathbb{Z}\times\mathbb{Z}/p\mathbb{Z}$. Now, using Definition 3, we can define a metric based for computational reasons on the 1-norm to compare musical bars together. We call it the DFT-distance, which we will use to extract a point cloud from a musical score.

Definition 4. *Let $\mathcal{B}$ and $\mathcal{B}'$ be two elements of $\mathfrak{B}_{t,p}$. The* ***DFT-distance*** *between $\mathcal{B}$ and $\mathcal{B}'$ is given by the 1-norm between their respective Fourier coefficients matrices:*

$$d_{\mathrm{DFT}}(\mathcal{B},\mathcal{B}') = \| \widehat{M}_{\mathcal{B}} - \widehat{M}_{\mathcal{B}'} \|_1 = \sum_{x=1}^{t}\sum_{y=1}^{p} |\mathcal{F}_{\mathcal{B}}(x,y) - \mathcal{F}_{\mathcal{B}'}(x,y)|$$

We denote by $(\mathfrak{B}_{t,p}, d_{\mathrm{DFT}})$ the metric space of all the musical bars of $\mathbb{Z}/t\mathbb{Z}\times\mathbb{Z}/p\mathbb{Z}$ equipped with the DFT-distance.

We can now turn any musical piece into a point cloud, i.e. a subset of a metric set. More precisely, if $\mathcal{S}_{\mathfrak{P}} = \{\mathcal{B}_1,\dots,\mathcal{B}_N\}$ is a musical score representing a musical piece $\mathfrak{P}$, then $\mathcal{S}_{\mathfrak{P}}$ is a subset of the metric space $(\mathfrak{B}_{t,p}, \mathrm{d}_{\mathrm{DFT}})$ and thus becomes a point cloud constructed in the following way:

- each **point** is a musical bar $\mathcal{B}_i$
- the **distance** between $\mathcal{B}_i$ and $\mathcal{B}_j$ is given by the DFT-distance.

Remark 1. In the following, we will apply a normalization on the distances computed with the DFT-distance in order to have all the barcodes on the same scale: in fact, our goal is to obtain barcodes that we can compare with each other, and we are more interested in the distance ratios between musical bars than in the distance values themselves. Therefore, by taking a percentage of the maximum distance for a given score, we consider the set of times of filtration to be $\{0,1,\dots,100\}$. Furthermore, we will say that we are looking at the filtration with a **scaling parameter** of $\epsilon\%$, with $\epsilon\in\{0,1,\dots,100\}$.

2.3 An Illustration of the Model

As an illustration of these definitions, let us consider the score $\mathcal{S}$ from Figure 6 which is an excerpt from the music piece *One Summer's Day*, composed by Joe Hisaishi in 2001 for the movie *Spirited Away*.

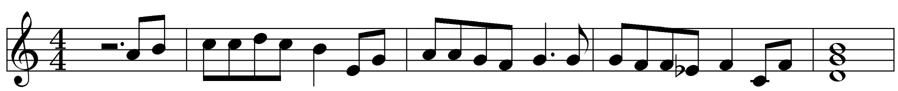


Fig. 6: The score $\mathcal{S}$ extracted from *One Summer's Day* written by Joe Hisaishi.

The above score is divided into five distinct musical bars: $\mathcal{S} = \{\mathcal{B}_1, \ldots, \mathcal{B}_5\}$. Furthermore, the shortest note here is a quaver one, so the time unit is $t = 8$, and the ambitus can be taken as $p = 24$, with $C_4 = 0 = C_6$, $C_5 = 12$ and $B_5 = 23$. With these observations, each musical bar $\mathcal{B}_i$ of the score $\mathcal{S}$ is a subset of $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/24\mathbb{Z}$. We extract the corresponding point cloud, as shown in Figure 7.

$\mathcal{B}_1 = \{(6,9),(7,11)\}$

$\mathcal{B}_2 = \{(0,12),(1,12),(2,14),(3,12),(4,11),(6,4),(7,7)\}$

$\mathcal{B}_3 = \{(0,9),(1,9),(2,7),(3,5),(4,7),(7,7)\}$

$\mathcal{B}_4 = \{(0,7),(1,5),(2,5),(3,3),(4,5),(6,0),(7,5)\}$

$\mathcal{B}_5 = \{(0,2),(0,7),(0,11)\}$

	$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}_3$	$\mathcal{B}_4$	$\mathcal{B}_5$
$\mathcal{B}_1$	0	86	81	82	63
$\mathcal{B}_2$		0	91	100	91
$\mathcal{B}_3$			0	95	87
$\mathcal{B}_4$				0	82
$\mathcal{B}_5$					0

Fig. 7: The point cloud associated with the score from Figure 6. The points are the musical bars in $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/24\mathbb{Z}$ (left) and the corresponding distance ratios are computed from the DFT-distance (right).

From this point cloud we can easily apply the Vietoris-Rips method, and the resulting filtration is presented in Figure 8. Notice that this illustration only shows the graphs filtration (to avoid overloading the figure): in particular, the triangles are all filled (always the case with the Vietoris construction).

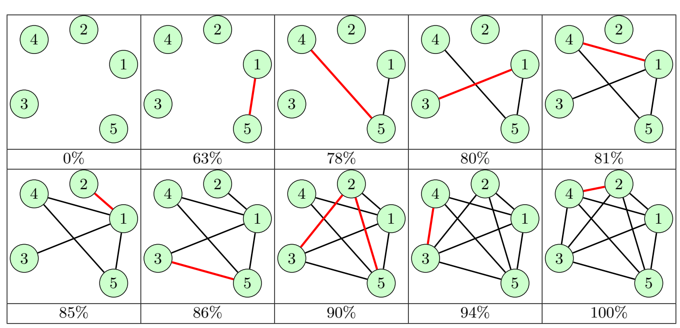


Fig. 8: The filtered simplicial complex associated with *One Summer's Day*.

For each scaling parameter $\epsilon \in \{0, 1, \ldots, 100\}$, the corresponding pairs of musical bars form edges. Therefore, we can look at the evolution of homology and then compute the associated barcodes family, as shown in Figure 9. Here and throughout this paper, we will focus only on barcodes in degree 0 and degree 1: indeed, the intuition of a musical interpretation goes along with these low dimensions, as for example we think of a cycle in dimension 1 as a musical loop. Of course, higher dimensions could be interpreted in future work. We can also note that there is no element of homology in degree 1, which is not so surprising considering the number of musical bars in the score.

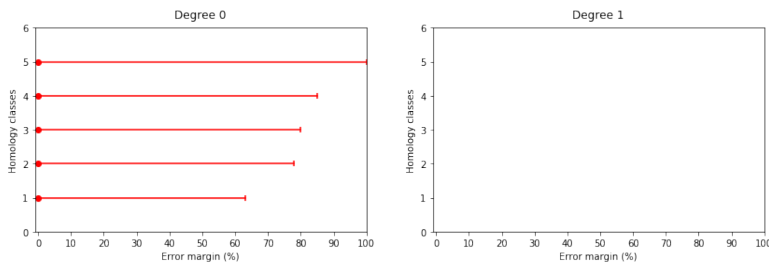


Fig. 9: The barcodes family associated with *One Summer's Day*.

3 Experiment on Artificial Scores

The goal of this section is to support the use of the two-dimensional DFT in the context of persistent homology, and in particular to answer the question of whether this new metric makes musical sense. To do this and in order to support this new model, we will try it on some artificially created musical scores: more precisely, we will construct a musical score consisting of very simple musical bars containing only one triad from a given family of chords, such as the major and minor chords (Euler's Tonnetz).

3.1 A Point Cloud from the Set of Minor and Major Chords

Let us consider the score based on the twenty-four major and minor chords, that means a score with twenty-four musical bars, each one containing a 3-chord, as shown in Figure 10. The point cloud thus contains twenty-four points, and can be identified with Euler's Tonnetz $T[3, 4, 5]$ of minor and major triads. A representation of this Tonnetz as a simplicial complex is given in Figure 11, and we refer to [5] for a formalized definition of the two-dimensional Tonnetze of the form $T[a, b, -(a + b)]$.

Notice that the representation from Figure 11 provides a topological structure for Euler's Tonnetz, which is given by a torus. We also see the action of the PLR-group of Neo-Riemannian transformations P (Parallel), L (Leading-tone) and R (Relative), which transforms a major (or minor) triad into the nearest minor (or major) triad. Note that this action is simply transitive, and we refer to [8] for more details on this subject. These two observations are exactly the kind of properties that we want to emphasize with our model. Indeed, it will support the use of the two-dimensional DFT in this particular context.

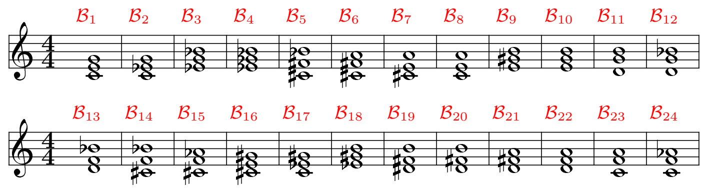


Fig. 10: The score of twenty-four musical bars based on the major and minor chords.

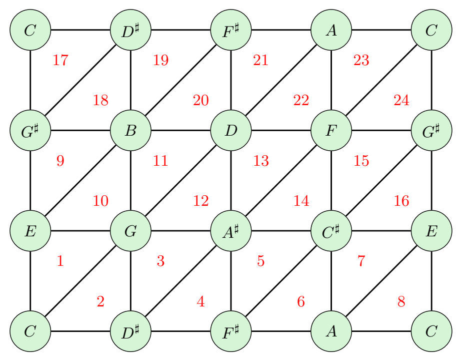


Fig. 11: Euler's Tonnetz as a simplicial complex $T[3, 4, 5]$.

3.2 The Results: PLR-Group and One-Dimensional Cycles

Now that we have the point cloud, we can compute the associated barcodes family, as presented in Figure 12.

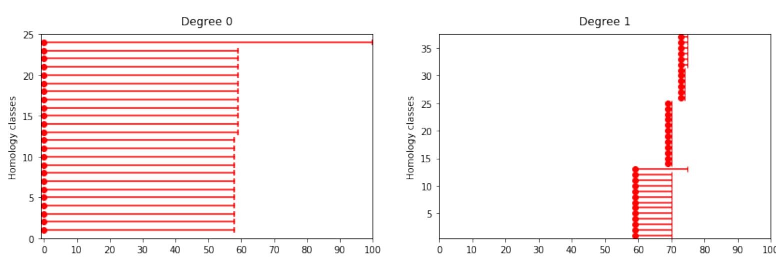


Fig. 12: The associated barcodes family with Euler's Tonnetz $T[3, 4, 5]$.

Let us analyze these two barcodes, starting with degree 0: at 58% of the filtration, each chord is connected to its relative, which means that the DFT understands this proximity. With a scaling parameter of 59%, the complex looks like in Figure 13 (left), where we have shown only the graphs (with vertices in $\{1, 2, \ldots, 24\}$ from the score 10): in this illustration, we have chosen to show the complexes at 59% and 69% because the filtration stagnates at 59% (the first moment where it is connected) and remains the same until it becomes as in 69%. There are several things to note here: first, in both cases we recover the shape of a torus, with a duality relation since a triangle in $T[3, 4, 5]$ is a vertex in our graph. Secondly, each vertex of the complex at 59% has exactly three neighbors: for instance, C-major triad (musical bar $\mathcal{B}_1$, vertex $\{1\}$) is connected to musical bars $\mathcal{B}_2$, $\mathcal{B}_8$ and $\mathcal{B}_{10}$. Looking back at the simplicial representation of Figure 11, we see that it corresponds to the minor chords C_m, A_m and E_m respectively, the three neighbors of C given by its parallel P, its relative R and its the leading-tone L, which are exactly the three minor chords that minimize the DFT-distance from a given triad. This observation is illustrated in Figure 14 and is formalized in Theorem 2.

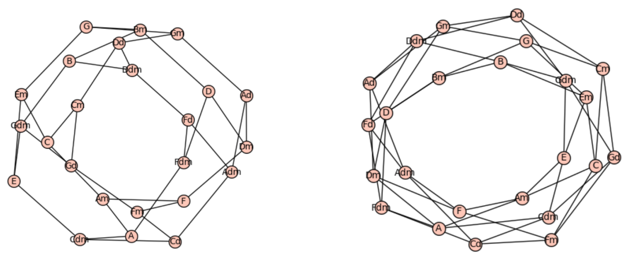


Fig. 13: The associated filtration with the Tonnetz $T[3, 4, 5]$ with a scaling parameter of 59% (left) and 69% (right).

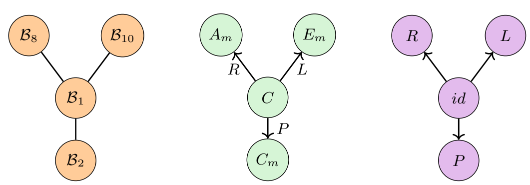


Fig. 14: A zoom on the graph at 59% of the filtration: each chord has exactly three neighbors that are given by the three basic transformations P, L and R.

Theorem 2. *The graph associated with the twenty-four major and minor chords of the Tonnetz* $T[3, 4, 5]$*, given by the filtration below with a scaling parameter of* 59%*, is exactly the Cayley graph of the* PLR*-group generated by the three transformations* P*,* L *and* R*.*

Proof. The action of the PLR-group on the set of the twenty-four major and minor chords is simply transitive, so there is a bijection between the elements of the PLR-group and the chords of the Tonnetz. Therefore, using the illustration from Figure 14, the left graph from Figure 13 is exactly the corresponding Cayley graph.

The barcode in degree 0 seems to capture the topological structure of the Euler's Tonnetz, and there is a strong relationship between the filtration and the PLR-group. On the other hand, the cycles in degree 1 also seem to emphasize this property: in fact, we can classify them into three types, all given by the transformation P, L and R. To illustrate these elements of 1-homology, we draw a representative of each type of one-dimensional cycle that we can construct from C-Major triad (but it is important to note that we can get these cycles from any chosen chord). These cycles are obtained respectively with transformations $(PRL)^2$, $(LP)^3$ and $(PR)^4$, as shown in Figure 15. Also notice that these cycles have been pointed out in [6] and it is musically consistent to recover such results.

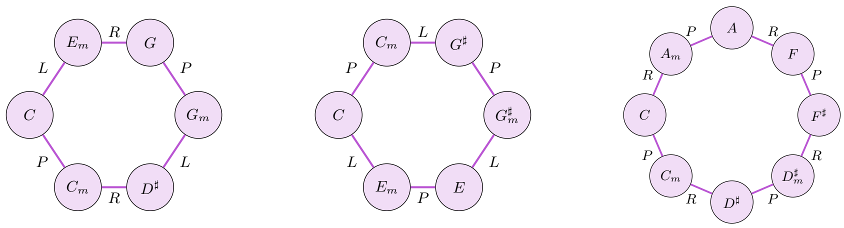


Fig. 15: The one-dimensional cycles obtain from the major and minor chords and the DFT as a metric: they are given repsectively by transformations $(PRL)^2$, $(LP)^3$ and $(PR)^4$ on a given chord (here C-Major).

3.3 A Study of the Two-Dimensional Tonnetze

As an extension of this work, we apply this new model to the eleven other sets of triads based on the two-dimensional Tonnetze of the form $T[a, b, -(a + b)]$. This analysis, which is fully described in [5], leads to a summary classification of the different filtrations for each point cloud, which is presented in Table 1.

The barcodes in degree 0 capture the topological structure of each Tonnetz. As for Euler's Tonnetz, we recover the shape of the different $T[a, b, -(a + b)]$ in the filtration, such as the tori, the cylinders, or even the several connected components for $T[2, 2, 8]$, $T[2, 4, 6]$, $T[3, 3, 6]$ and $T[4, 4, 4]$. In addition, barcodes of degree 1 allow us to manually generate one-dimensional cycles. Each Tonnetz provides specific types of generators of H_1, and it is interesting to compare them: for some Tonnetz we have cycles of length 4, while for others it can go up to 12. Table 1 classifies the different types of cycles we can find in these two-dimensional Tonnetze.

4 Conclusion and Perspective for Future Work

We begin this paper by introducing a new method for extracting a filtration and more precisely a point cloud from a given score. In order to support this approach, we wanted to consider the inverse problem, that is, to apply the DFT together with persistent homology to artificially constructed musical scores and show that the DFT is a reasonable choice of metric.

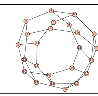
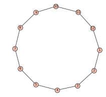
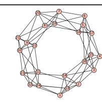
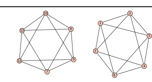
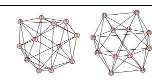
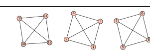
Point cloud	Topology	Most relevant stage of the filtration	Length of cycles (degree 1)				
			4	6	8	10	12
$T[1,2,9]$, $T[2,3,7]$	Torus			×	×		×
$T[1,3,8]$, $T[3,4,5]$				×	×		
$T[1,4,7]$				×			
$T[1,1,10]$, $T[2,5,5]$	Cylinder						×
$T[1,5,6]$	Necklace of six tetrahedra					×	
$T[2,2,8]$	Two cylinders			×			
$T[2,4,6]$	Two necklaces of three tetrahedra		×	×			
$T[3,3,6]$	Three tetrahedra		×				
$T[4,4,4]$	Four triangles						

Table 1: A classification of the twelve two-dimensional Tonnetz $T[a, b, -(a+b)]$ by their topological structure and their types of cycle in H_1 using the DFT.

The score we choose is given by triads from a specific space, specifically the Euler's Tonnetz, or more generally any two-dimensional Tonnetz of the classical form $T[a, b, -(a+b)]$. The barcodes in degrees 0 and 1 seem to reveal consistent musical features: in particular, we find back the topological structure associated with the Tonnetze, such as the torus in the case of $T[3, 4, 5]$. This construction also allows us to emphasize the strong relationship with PLR-group of fundamental transformations of Euler's Tonnetz, as illustrated by Theorem 2. Finally, we recover some famous cycles with barcode in degree 1, which is musically worthwhile. This confirms that the metric constructed by means of the DFT, together with persistent homology, seems to be a reasonable tool for understanding known musical structures.

However, there are several other artificial musical scores on which we might want to test our model. As a natural starting point, we could try artificial scores based on scales, as in [5]. We could also try the metric on the set of all the triads in $\mathbb{Z}/12\,\mathbb{Z}$: for a given triad, the closest would still be its relative, then its leading-tone and its parallel, but this will allow us to classify all the 220 triads with this DFT, and this will also give a meta-classification between the twelve Tonnetze. Another work could be to compare the DFT with other metrics, such as the Hausdorff distance which provide much less interesting results (see [5]). Finally, the DFT-distance could also be applied to n-chords, or simply to higher dimensional Tonnetze, as presented in [9]. To go further, we could also imagine trying out our metric on musical objects that focus more on rhythms, such as the general *Zeitnetz* that is introduced in [11].

Finally, we have proposed here a new model for associating a filtration of simplicial complexes with a musical score, and thus a topological signature given by barcodes. If this method seems to provide strong and encouraging results in applications on artificial scores, we also apply it as a natural work in the context of more general problems, such as automatic classification of musical styles. These other kinds of applications of this model based on the DFT and persistent homology are presented in [5].

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